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Nontriviality of $SK_1(R[M])$

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Abstract

The main result of this paper is as follows:

For any commutative regular (actually even K_2-regular) ring R and any finitely generated intermediate monoid $\mathbb{Z}^r_+ \subset M \subset \mathbb{Q}^r_+$ (for some natural r) the following conditions are equivalent:

(a) $M \approx \mathbb{Z}^r_+$,

(b) $R[M]$ is K_1-regular,

(c) M is seminormal and $SK_1(R) = SK_1(R[M])$ (i.e. the natural homomorphism $SK_1(R) \to SK_1(R[M])$ is an isomorphism),

and, if in addition $\Omega_R \neq 0$,

(d) $SK_1(R) = SK_1(R[M])$,

where Ω_R is a module of absolute differentials. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are well known.

In Sections 8–10 we present examples, further generalizations and applications.

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1. Introduction

At first we should like to mention some words on motivations of the results presented below. In our previous papers [10–14], concerning K-theoretical properties of commutative monoid rings, we established some properties of such rings $R[M]$ from the point of view of which the extensions of type $R \subset R[M]$ are similar to the "classical" case of polynomial extensions $R \subset R[t_1, \dots, t_r]$, for which a "typical" theorem asserts that $F(R) = F(R[t_1, \dots, t_r])$, F being some K-theoretical functor. For instance, in case $F = K_1$ this is done in [6]. This paper deals with those K-theoretical properties, which distinguish the extensions $R \subset R[M]$ with M nonfree from $R \subset R[t_1, \dots, t_r] = R[\mathbb{Z}^r_+]$, where $\mathbb{Z}_+$ denotes the additive monoid of nonnegative integers. Let us mention some previous results in this direction. In [17] Srinivas constructed the first explicit example of a nonfree normal monoid M which is integrally embedded in $\mathbb{Z}^2_+$ such that $SK_1(k[M]) \neq 0$, where k is any algebraically

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closed field of characteristic $\neq 2$. The monoid was the one generated by $(2,0)$, $(1,1)$ and $(0,2)$. It should be noted here that for an integral monoid extension $N \subset \mathbb{Z}_+^r$ ($r \in \mathbb{N}$) there always exists an intermediate monoid $\mathbb{Z}_+^r \subset L \subset \mathbb{Q}_+^r$ isomorphic to N (see Proposition 2.2), where $\mathbb{Q}_+$ denotes the additive monoid of nonnegative rational numbers. In [12] we could show that for any commutative regular ring R there exist infinitely many finitely generated normal sub-monoids $M \subset \mathbb{Z}_+^2$, for which $SK_1(R) \neq SK_1(R[M])$. However our arguments from [12, Section 9] did not allow us to indicate explicit representatives of such monoids. We remark here that it was just the lack of the excision property for K_1 that yielded such counterexamples to the direct K_1-analogue of Anderson's conjecture [2, 3, 10]. In [11] we proved that $SK_1(R) = SK_1(R[M])$ for all regular rings R whenever the (commutative, cancellative, torsion free) monoid M is c divisible for some natural $c \geq 2$ and the group of units of M is trivial. Later in [14] we showed (basing on Suslin–Wodzicki's results on excision in algebraic K-theory) that excision in K-theory of such monoid rings (i.e. corresponding to c divisible monoids) holds. This observation allowed us to establish isomorphisms of type $K_i(R) = K_i(R[M])$ for all $i \geq 0$ and all c divisible ($c \geq 2$) submonoids $M \subset \mathbb{Q}_+^r$ ($r \in \mathbb{N}$) for which the extension $M \subset \mathbb{Q}_+^r$ is integral (see Definition 2.1).

The results of the present paper show that Bass–Heller–Swan's classical result on K_1-regularity of a regular ring [6] essentially does not admit the natural generalization to the situation of monoid ring extensions $R \subset R[M]$ with M nonfree and finitely generated. Our approach is completely different from that of [17] and does not make use of the advanced technique involved there. Our arguments are based on the results of [7] and we explicitly construct nontrivial elements in $SK_1(R[M])$. In particular, we derive the same nonzero element of $SK_1(k[X^2, XY, Y^2])$ as in [17].

To conclude this introduction it seems to be a good place to set the following

Question: Is our main result valid for all finitely generated (commutative, cancellative, torsion free) monoids without nontrivial units?

In Section 9 we indicate a much wider class of finitely generated monoids (than the one of intermediate monoids $\mathbb{Z}_+^r \subset M \subset \mathbb{Q}_+^r$), which also can be attacked by our methods.

2. Φ-correspondence

For the readers convenience in each of our previous papers [10–14] we included the detailed description of Φ-correspondence. Wishing to make the present paper self-contained we attach to it one more such a description (without detailed proofs). In this section the monoid operation will be written additively.

All monoids and rings we consider are assumed to be *commutative*. The monoids we deal with are also *cancellative* and *torsion free*. For a monoid M its group of quotients will be denoted by $K(M)$. $U(M)$ will denote the group of units of M. Due to our conditions for a monoid M we have the natural embeddings

$$M \rightarrowtail K(M) \rightarrowtail \mathbb{Q} \underset{\mathbb{Z}}{\otimes} K(M) \approx \bigoplus_r \mathbb{Q}$$

(for some cardinal r), where $\mathbb{Z}$ and $\mathbb{Q}$ denote the additive groups of integers and rationals, respectively; r will be called the rank of M (rank(M)). By $\mathbb{Z}_+$ and $\mathbb{Q}_+$ will be denoted the additive monoids of nonnegative integers and nonnegative rationals, respectively. For a monoid M we fix one of the embeddings $M \rightarrowtail \bigoplus_r \mathbb{Q}$ obtained by the composite map $M \rightarrowtail K(M) \rightarrowtail \mathbb{Q} \otimes_{\mathbb{Z}} K(M) \approx \bigoplus_r \mathbb{Q}$. In what follows, we always assume that $r < \infty$. Denote by $C(M)$ the convex cone in $\bigoplus_r \mathbb{R}$ (r-dimensional real space) with vertex in the origin $\mathrm{O} \in \bigoplus_r \mathbb{R}$, which is spanned by M. Let S^{r-1} denote the standard unit sphere in $\bigoplus_r \mathbb{R}$ and put $\Phi(M) = C(M) \cap S^{r-1}$. Then $\Phi(M)$ will turn out to be a *convex* subset of S^{r-1}, where the notion of convexity in S^{r-1} is introduced as follows: a subset $X \subset S^{r-1}$, is called convex if for arbitrary $x, y \in X$, which are not opposite on S^{r-1}, the unique shortest line in S^{r-1}, connecting x and y, is contained in X. Let M^+ denote the set $M \backslash \{0\}$. For $m \in M^+$ we put $\Phi(m) = \{$the point of intersection of the radial direction of m with $S^{r-1}\}$. Observe that $\Phi(M)$ coincides with the convex hull in S^{r-1} of $\{\Phi(m) \mid m \in M^+\}$ (= the smallest convex set containing $\{\Phi(m) \mid m \in M^+\}$).

A (convex) *polyhedron* in S^{r-1} is defined as an intersection of some finite system of (closed) hemispheres of S^{r-1}. A polyhedron $P \subset S^{r-1}$ will be called *finite* if it does not contain a pair of opposite points. A finite polyhedron is always contained in the interior of some hemisphere. A monoid M without nontrivial units is finitely generated if and only if $K(M)$ is finitely generated and $\Phi(M)$ is a finite polyhedron (the "if" part of this statement is equivalent to Gordan's classical lemma, see [11–14]).

The points of type $\Phi(x)$ for some $x \in (\mathbb{Q}_+^r)^+$ will be called rational. Clearly, $\{\Phi(m) \mid m \in M^+\}$ is a dense subset of $\Phi(M)$, consisting of all rational points of it.

A *simplex* $\Delta \subset S^{r-1}$ is defined as a finite polyhedron the dimension of which $+ 1$ is equal to the number of its vertices. A *pyramid* $\delta \subset S^{r-1}$ with a chosen vertex z is a convex hull of $\{z\} \cup X$ for some finite subset $X \subset S^{r-1}$, such that this hull is a finite polyhedron and $\dim(\delta) - 1$ is equal to the dimension of the convex hull of X.

Let M be a monoid and W a convex subset of $\Phi(M)$ of arbitrary dimension. Then we put $M(W) = \{m \in M^+ \mid \Phi(m) \in W\} \cup \{0\}$. Clearly, $M(W)$ is a submonoid of M. We shall also use the following notation $M_* = M$(relative interior of the convex set $\Phi(M)$), where a relative interior of some convex subset X of a (real) affine space is defined as the largest open set of the affine hull of X in the ambient space, which is contained in X. In particular, an interior of a point is the point itself. Hence in case rank$(M) = 1$ we have $M_* = M$.

Definition 2.1. (a) For an extension of monoids $M \subset N$ an element $n \in N$ is called integral over M if some positive multiple of n belongs to M,

(b) For an extension of monoids $M \subset N$ the monoid M is called normal in N if there is no integral element over M in $N \backslash M$,

(c) a monoid M is called normal if M is normal in $K(M)$,

(d) a monoid M is called seminormal if $x \in K(M)$, $2x \in M$, $3x \in M$ imply that $x \in M$.

Proposition 2.2. (a) *Let M be a monoid and $W \in \Phi(M)$ a convex subset. Then $M(W)$ is normal in M; a submonoid $N \subset M$ which is normal in M is normal or seminormal whenever M is so, respectively,*

(b) *let M and W be as above and* $\dim(W) = \dim(\Phi(M))$, *then* $K(M(W)) = K(M)$,

(c) *let M be a finitely generated normal monoid with trivial $U(M)$; assume that P is a vertex of the finite polyhedron $\Phi(M)$, then $M(P)^{-1}M = \{m - n \mid m \in M, n \in M(P)\} \approx \mathbb{Z} \oplus M'$ for some finitely generated normal monoid M' with trivial $U(M')$,*

(d) *for a seminormal monoid M the submonoid $M_* \subset M$ is a normal monoid,*

(e) *for a monoid M the set $\Phi(M)$ is a simplex if and only if M is isomorphic to a submonoid $N \subset \mathbb{Q}^r_+$, $r = \operatorname{rank}(M)$, for which the extension $N \subset \mathbb{Q}^r_+$ is integral.*

(f) *let M be a finitely generated monoid; then $\Phi(M)$ is a simplex if and only if M is isomorphic to a finitely generated intermediate monoid $\mathbb{Z}^r_+ \subset N \subset \mathbb{Q}^r_+$, $r = rank(M)$, and if and only if M is isomorphic to a submonoid $L \subset \mathbb{Z}^r_+$, $r = rank(M)$ for which the extension $L \subset \mathbb{Z}^r_+$ is integral.*

Proof. (a) is trivial; (b), (e) and (f) are considered in [12]; (c) and (d) are proved in [10]. □

Notations 2.3. For a monoid M we put

(a) $sn(M) = \{x \in K(M) \mid cx \in M$ for all sufficiently large natural $c\}$ (seminormalization of M),

(b) $n(M) = \{x \in K(M) \mid cx \in M$ for some natural $c\}$ (normalization of M),

(c) in case M is normal for an arbitrary convex subset $W \subset S^{r-1}$, $r = \operatorname{rank}(M)$, $M(W) = K(M)(W)$.

Remarks. $sn(M)$ is the smallest seminormal monoid containing M and $n(M)$ is the smallest normal monoid containing M; further, notation (c) above is compatible with our earlier notation $M(W)$, because if M is normal and $W \subset \Phi(M)$ both these notations denote the same objects.

Proposition 2.4. *For a finitely generated monoid M with trivial $U(M)$ the monoids $sn(M)$ and $n(M)$ are also finitely generated, without nontrivial units and satisfy the equalities*

$$sn(M)_* = sn(M_*) = n(M)_* = n(M_*).$$

So if M is seminormal then $M_* = n(M)_*$.

Proof. The proof follows easily from 2.2 (b) and (d). □

Proposition 2.5. *Let M be a finitely generated normal monoid with trivial $U(M)$. Then M is isomorphic to a submonoid $N \subset \mathbb{Z}^r_+$, $r = \mathrm{rank}(M)$, which is normal in $\mathbb{Z}^r_+$.*

Proof. See [14, Part 1]. □

Proposition 2.6. *Let M be a finitely generated monoid of rank ≥ 2, $U(M) = 0$ and P be a vertex of $\Phi(M)$. Then there exists a submonoid $M' \subset M$, which is normal in M, $M'_* \subset M_*$ and $M(P)^{-1}M = K(M(P)) \oplus M'$.*

Proof. By Proposition 2.2(c) $M(P)^{-1}M = K(M(P)) \oplus \hat{M}$ for some submonoid $\hat{M} \subset K(M)$, which is normal in $K(M)$ (here we first have to observe that $U(M(P)^{-1}M) = K(M(P)))$. By Proposition 2.5 there exists a monoid extension $\hat{N} \subset \mathbb{Z}^{r-1}_+$ such that $\hat{N}$ is normal in $\mathbb{Z}^{r-1}_+$ and $\hat{N} \approx \hat{M}$ ($r = \mathrm{rank}(M)$). By Proposition 2.2(b) $K(\hat{N}) = \mathbb{Z}^{r-1}$. This means that there exists a free basis $\{x_1, \ldots, x_{r-1}\}$ of $K(\hat{M}) \approx \mathbb{Z}^{r-1}$, such that $\hat{M}$ is normal in the free monoid F spanned by $\{x_1, \ldots, x_{r-1}\}$. For any natural number c consider the following system of elements $\{x_1 + cx_r, \ldots, x_{r-1} + cx_r\}$, where x_r is a generator of $M(P) \approx \mathbb{Z}_+$. We claim that for c sufficiently large $M(P)^{-1}M = K(M(P)) \oplus M_c$, $M_c \subset M$, M_c is normal in M and $(M_c)_* \subset M_*$, where $M_c = M \cap F_c$ for the free monoid F_c spanned by $\{x_1 + cx_r, \ldots, x_{r-1} + cx_r\}$. Let Δ_c be the simplex $\Phi(F_c)$. When c is sufficiently large all the vertices of Δ_c are sufficiently close to P and, simultaneously, these vertices belong to the geodesics (of S^{r-1}), spanned by the pairs $\{P, \Phi(x_1)\}, \ldots, \{P, \Phi(x_{r-1})\}$, respectively. Now consider an arbitrary decomposition $\Phi(M) = \delta \cup \gamma$, where δ is a pyramid with vertex P and γ is a finite polyhedron meeting δ at its base ($\dim(\delta) = \dim(\Phi(M))$). The equality $M(P)^{-1}M = K(M(P)) \oplus \hat{M}$ implies that the aforementioned decomposition $\Phi(M) = \delta \cup \gamma$ can be chosen so that δ will be included in the pyramid $\Phi(M(P) \oplus \hat{M})$. In this situation each face of δ, which passes through P, will be contained in some face of $\Phi(M(P) \oplus \hat{M})$, passing through P (an easy geometrical observation). Arbitrarily fix such a decomposition $\Phi(M) = \delta \cup \gamma$. We easily see that for c sufficiently large $\dim(\Delta_c \cap \mathrm{int}(\delta)) = r - 2$ (int is for interior). This means that for c sufficiently large $M_c = M \cap F_c = M(\Delta_c \cap \delta)$ is of rank $r - 1$ and $(M_c)_* \subset M_*$ (the normality of M_c in M is obvious). It only remains to show that $K(M(P)) \oplus M_c = K(M(P)) \oplus \hat{M}$. By easy geometrical observations both $\Phi(M(P)) \oplus M_c)$ and $\Phi(M(P)) \oplus \hat{M})$ coincide with the convex hull of $\delta \cup \Phi(-x_r)$. By normality of $K(M(P)) \oplus M_c$ and $K(M(P)) \oplus \hat{M}$ we shall be done whenever the coincidence of the quotient groups of these two monoids will be established. We have $K(K(M(P)) \oplus M_c) = K(M(P)) \oplus K(M_c) = K(M(P)) \oplus K(F_c)$, since by Proposition 2.2(b) $K(M_c) = K(F_c)$. Finally, we get $K(M(P)) \oplus K(F_c) = K(M(P)) \oplus K(F) = K(M(P)) \oplus K(\hat{M}) = K(K(M(P)) \oplus \hat{M})$. □

Definition 2.7. A monoid M will be called Φ-simplicial if M is finitely generated, $U(M) = 0$ and $\Phi(M)$ is a simplex.

Proposition 2.8. *Let M be a Φ-simplicial monoid and P a vertex of $\Phi(M)$. Assume the following conditions are satisfied for M and P:*

(i) *for each codimension 1 face d of $\Phi(M)$, passing through P, the monoid $M(d)$ is free,*

(ii) *there exists a free submonoid $F \subset M$ for which P is one of the vertices of $\Phi(F)$, each proper face of $\Phi(F)$ passing through P is contained in some proper face of $\Phi(M)$ passing through P and $K(F) = K(M)$.*

Then M is free whenever $r = \operatorname{rank}(M) \geq 3$.

Proof. Let $\{P, Q_1, \dots, Q_{r-1}\}$ be the set of all vertices of $\Phi(M)$. By (i) the submonoids $M(P), M(Q_1), \dots, M(Q_{r-1}) \subset M$ are all isomorphic to $\mathbb{Z}_+$. Denote by $t, t_1, \dots, t_{r-1}$ the corresponding generators. We claim that M is generated by $\{t, t_1, \dots, t_{r-1}\}$ (in this situation M is obviously free). Let L be the free monoid, spanned by $\{t, t_1, \dots, t_{r-1}\}$. We have the inclusions $L \subset M \subset n(M)$. So it will suffice to show $L = n(M)$. Since $\Phi(L) = \Phi(M) = \Phi(n(M))$ and L and $n(M)$ are normal we only have to show that $K(L) = K(n(M))$. But $K(n(M)) = K(M)$. Thus our claim reduces to the equality $K(L) = K(M)$. Let $\{t, s_1, \dots, s_{r-1}\}$ be a basis of $F \approx \mathbb{Z}_+^r$. Clearly we can assume that $\Phi(s_i) \in [\Phi(t_i), \Phi(t)]$ (the convex hull of $\{\Phi(t_i), \Phi(t)\}$), $i \in [1, r-1]$. For $i \in [1, r-1]$ let d_i be the codimension 1 face of $\Phi(M)$, which does not pass through Q_i ($P \in d_i$). Put $d_i' = d_i \cap \Phi(F)$. Let us show that $K(M(d_i)) = K(F(d_i'))$. By Proposition 2.2(b) it will suffice to show that $F(d_i')$ is normal in $M(d_i)$ (observe that $\dim(d_i) = \dim(d_i') = r-2$). Equivalently, it will suffice to show that $M(d_i') = F(d_i')$. Assume $x \in M(d_i') \backslash F(d_i')$. By (ii) $x \in K(F)$. On the other hand, x is integral over $F(d_i')$ (since $\Phi(x) \in d_i'$) and, hence, over F. By normality of F we get $x \in F$, a contradiction because $F \cap M(d_i') = F(d_i')$. Thus, $K(M(d_i) = K(F(d_i')$ for each $i \in [1, r-1]$. Then we have the following sequence of monoids, having the same groups of quotients,

$$F \subset F_1 \subset \cdots \subset F_{r-1} = L,$$

where for each $i \in [1, r-1]$ F_i is the free monoid, spanned by $\{t, t_1, \dots, t_i, s_{i+1}, \dots, s_{r-1}\}$. In particular, $K(F) = K(L)$. But $K(F) = K(M)$. $\square$

Below we shall also need

Proposition 2.9 (Gubeladze [12, Lemma 5.3]). *For a finitely generated monoid M with trivial $U(M)$ there exists an element $m \in M_*^+$ for which $m + n(M) \subset M$.*

Actually Lemma 5.3 of [12] states that there exists an element $m \in M$ for which $m + n(M) \subset M$, but then for an arbitrary element $m' \in M_*^+$ the element $m + m'$ satisfies the desired condition.

Later on for a finitely generated monoid without nontrivial units we shall use the notation $\operatorname{int}(M) = M_*^+$. Hence $\operatorname{int}(M)$ consists of those nonzero elements of M the radial directions of which pass through the interior of $\Phi(M)$. In particular, if $\operatorname{rank}(M) = 1$ then $\operatorname{int}(M)$ is just the set of all nonzero elements of M (because a point itself is its interior).

3. Rank 2 normal monoids

Since 1-dimensional finite polyhedra are all simplices by Proposition 2.2(f) all finitely generated rank 2 monoids M with trivial $U(M)$ can be embedded in $\mathbb{Z}_+^2$ so that the extensions $M \rightarrowtail \mathbb{Z}_+^2$ will be integral. In case M is normal the embedding can be described more explicitly:

Proposition 3.1. *Let M be a nonfree finitely generated normal monoid of rank 2 with trivial $U(M)$. Then there exist natural numbers j and n satisfying the conditions*

(i) *j and n are coprime and $0 < j < n$,*

(ii) *M is isomorphic to the submonoid $N \subset \mathbb{Z}_+^2$, generated by the elements $(n,0),(1,j),(2,\overline{2j}),\dots,(n-1,\overline{(n-1)j}),(0,n)$, where "bar" denotes the remainder with respect to the division by n.*

Observe that the extension $N \subset \mathbb{Z}_+^2$, mentioned in the proposition is integral.

This proposition was proved by Anderson in [1, Theorem 2.5]. Actually Anderson proved a structural theorem for affine normal 2-dimensional subrings of $k[X,Y]$, generated by monomials, but Proposition 3.1 is essentially equivalent to the Anderson's mentioned result because the k-algebra isomorphisms between monomial algebras considered in [1] are actually induced by monomial transformations.

Now, using Proposition 3.1 we shall show that the normal embedding $M \subset \mathbb{Z}_+^r$, $r = \operatorname{rank}(M)$, mentioned in Proposition 2.5, can be chosen in a very special way when $r = 2$.

Proposition 3.2. *Let M be a finitely generated nonfree rank 2 normal monoid with trivial $U(M)$. Then M is isomorphic to a submonoid $L \subset \mathbb{Z}_+^2$, which satisfies the conditions:*

(i) *L is normal in $\mathbb{Z}_+^2$,*

(ii) *$(0,1) \in L$, $(1,1) \in L_*$, $(1,0) \notin L$.*

Proof. By Proposition 3.1 we can assume that M is a submonoid of $\mathbb{Z}_+^2$, generated by elements $(n,0)$, $(1,j)$, $(2,\overline{2j}),\dots,(n-1,\overline{(n-1)j}),(0,n) \in \mathbb{Z}_+^2$ for some coprime natural j and n. For $i \in [1,n]$ suppose $ij = nq_i + \overline{ij}$. We define the monoid homomorphism $\Psi : \mathbb{Z}^2 \to \mathbb{Q}^2$ by $(0,1) \mapsto (0,1/n)$ and $(1,0) \mapsto (1, 1 - j/n)$. We have

$$(0,n) \overset{\Psi}{\longmapsto} (0,1),$$

$$(1,j) \longmapsto (1,1),$$

$$(i,\overline{ij}) \longmapsto (i, i - q_i) \quad \text{for } i \in [2,n].$$

Since $ij \geq q_i n$ and $j < n$ we have $i - q_i > 0$. Hence the image of M is contained in $\mathbb{Z}_+^2$. Denote by L this image. Clearly $M \approx \dot{L}$. The verification of condition (ii) for L is straightforward. To obtain the normality of L in $\mathbb{Z}_+^2$ it suffices to observe that

$K(L) = \mathbb{Z}^2$ (since $(0,1) \in L$ and $(1,1) - (0,1) = (1,0) \in K(L)$) and L is normal (being the isomorphic image of M). Hence condition (i). $\square$

4. Swan's elements in $K_2(R[X, Y]/(XY))$

For a ring A and two elements $a, b \in A$, such that $1 + ab \in U(A)$ (multiplicative group of units) Dennis and Stein in [8] defined the element $\langle a, b \rangle \in K_2(A)$ (Milnor K-group of A). The simplest case of the described situation is $A = R[X, Y]/(XY)$ and $a = u$, $b = v$, where R is some ring, X and Y are variables and u and v denote the images of X and Y in A, respectively. In this situation $\langle a, b \rangle$ coincides with Swan's element $[x_{12}(u), x_{21}(v)] \in K_2(R[X, Y]/(XY))$, where $[-,-]$ denotes a commutator and $x_{ij}(-)$ are standard generators of a Steinberg group [19].

Proposition 4.1. *Let R, u and v be as above. Then*
(a) $\langle u, v \rangle \notin K_2(R)$, *where $K_2(R)$ is identified with its isomorphic image in $K_2(R[u, v])$*,
(b) $\langle u^a, v^b \rangle = [x_{12}(u^a), x_{21}(v^b)] = 0$ *whenever* $\max(a, b) \geq 2$.
(c) $\langle a, 0 \rangle = \langle 0, a \rangle = 0$ *for an arbitrary ring A and arbitrary element $a \in A$.*

Proof. (a) immediately follows from the results of [7], (b) is a consequence of the relation $D3$ and (c) that of $D2$ from [7, Section 1]. $\square$

Theorem 4.2. *Let R be a K_2-regular ring and $R[u, v]$ as above. Then $K_2(R[u, v]) = \langle u, v \rangle R \oplus K_2(R)$, where $\langle u, v \rangle R$ is the free (rank 1) R-module, generated by $\langle u, v \rangle$.*

This theorem exactly coincides with Corollary 4.8 from [7].

Remark. The R-module stucture on $K_2(R[u, v])/K_2(R) \approx \langle u, v \rangle R$ is defined by $r\langle u, v \rangle = \langle ru, v \rangle$. More generally, for any graded ring A with A_0 in degree 0, the relative groups $K_i(A)/K_i(A_0)$ for arbitrary natural i are naturally modules over the Witt vectors $W(A_0)$. This is proved in [24, p. 468].

Below we shall identify $\mathbb{Z}_+^2$ with the multiplicative monoid of monomials $\{X^a y^b \mid a, b \in \mathbb{Z}_+\}$.

Let R be a (commutative) ring and let X, Y, u, v be as above. Consider the following natural group homomorphisms:

$$\begin{tikzcd} St(R[X, Y]) \arrow[r] \arrow[d] & E(R[X, Y]) \\ St(R[u, v] & \end{tikzcd}$$

Proposition 4.3. *For an arbitrary convex neighborhood* W of $\Phi(XY)$ *in* $\Phi(\mathbb{Z}_+^2)$ *there exists a lifting* $z \in St(R[X,Y)$ *of* $\langle u,v\rangle \in K_2(R[u,v])$ *the image* A *of which in* $E(R[X,Y])$ *belongs to* $SL(R[\mathbb{Z}_+^2(W)]) \cap SL(R[X,Y],(XY))$.

Proof. Consider the following system of preimages of $\langle u,v\rangle \in K_2(R[u,v])$ in $St(R[X,Y]$:

$$z_1 = [x_{12}(X), x_{21}(Y)],$$

$$z_{i+1} = x_{12}(X^{i+1}Y^i)z_i x_{21}(-X^iY^{i+1}), \quad i \ge 1.$$

Let A_i denote the image of z_i in $E(R[X,Y])$. Observe for each i the matrix A_i has a form

$$\begin{bmatrix} f_i & -X^{i+1}Y^i \\ X^iY^{i+1} & 1-XY \end{bmatrix} \in SL_2(R[X,Y]),$$

where $f_i \in \sum_{j=0}^{j=2i}(XY)^j$. Indeed

$$A_1 = \begin{bmatrix} 1+XY+X^2Y^2 & -X^2Y \\ XY^2 & 1-XY \end{bmatrix}$$

and

$$A_i = \begin{bmatrix} f_{i-1}+X^{2i-1}Y^{2i-1}+X^{2i}Y^{2i} & -X^{i+1}Y^i \\ X^iY^{i+1} & 1-XY \end{bmatrix}$$

for $i \ge 2$. Hence the induction process applies. Denote by W_i the subsegment of $\Phi(\mathbb{Z}_+^2)$, spanned by $\Phi(X^iY^{i+1})$ and $\Phi(X^{i+1}Y^i)$. It is obvious that each W_i is a (convex) neighborhood of $\Phi(XY)$, $W_i \supset W_{i+1}$ for $i \ge 1$ and $\bigcap_i W_i = \Phi(XY)$. Since $A_i \in SL(R[\mathbb{Z}_+^2(W_i)]) \cap SL(R[X,Y],(XY))$ the element z_i with sufficiently large i satisfies the desired condition. □

Now we are ready to turn to the proof of our main result. The proof occupies next three sections.

5. The case of rank 2 seminormal monoids

Proposition 5.1. *Let* R *be a* K_2*-regular (commutative) ring and* M *a non-free finitely generated seminormal rank* 2 *monoid with trivial* $U(M)$. *Then the image of the natural homomorphism* $SK_1(R[M_*]) \to SK_1(R[M])$ *is strictly larger than* $SK_1(R)$.

Here $SK_1(R)$ is identified with its isomorphic image in $SK_1(R[M])$.

We see that Proposition 5.1 implies the special case of our main result for rank 2 seminormal monoids.

Proof. *Step* 1: First we consider the case when M is normal. By Proposition 3.2 we can assume that $M \subset \mathbb{Z}_+^2$, M is normal in $\mathbb{Z}_+^2$, $Y \in M$, $XY \in M_*$ and $X \notin M$. Since $\Phi(XY)$ is an internal point of $\Phi(M)$ we get that $\Phi(M_*)$ is a convex neighborhood of $\Phi(XY)$ in $\Phi(\mathbb{Z}_+^2)$. By Proposition 4.3 there exists a preimage $z \in St(R[X, Y])$ of $\langle u, v\rangle \in K_2(R[u, v])$, for which the corresponding image A in $E(R[X, Y])$ belongs to $SL(R[M_*]) \cap SL(R[X, Y]), (XY))$. Let $[A]$ denote the image of A in $\mathrm{SK}_1(R[M])$. We claim that $[A] \notin SK_1(R)$. Suppose $[A] \in SK_1(R)$. Then by the augmentation $R[X, Y] \to R$, for which $X, Y \mapsto 0$, we would have $[A] = 0$ (in $SK_1(R[M])$), or equivalently $A \in E(R[M])$. Assume s is some preimage of A in $St(R[M])$, s_1 is the image of s in $St(R[X, Y])$ and s_2 is the image of s_1 in $St(R[u, v])$. Since both s_1 and z map into $A \in E(R[X, Y])$ we have $s_1 z^{-1} \in K_2(R[X, Y]) = K_2(R)$ (by K_2-regularity). Therefore, $s_2\langle u, v\rangle^{-1} \in K_2(R)$. Modifying the choice of s by this element from $K_2(R)$ we can assume that actually $s_2 = \langle u, v\rangle^{-1}$. On the other hand, the image of $R[M]$ in $R[u, v]$ under the composite map $R[M] \to R[X, Y] \to R[u, v]$ is $R[v]$ (because of our conditions on the embedding $M \subset \mathbb{Z}_+^2$). We get $\langle u, v\rangle \in St(R[v])$. Here $St(R[v])$ is identified with its natural isomorphic image in $St(R[u, v])$ with respect to the R-retraction $R[v] \rightleftarrows R[u, v]$, $u \mapsto 0$. Now the following commutative diagram:

$$\begin{tikzcd}
St(R[u,v]) \arrow[rr, "u \mapsto 0"', "v \mapsto v"] & & St(R[u,v]) \\
& St(R[v]) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

implies that $\langle u, v\rangle = \langle 0, v\rangle = 0$ (by Proposition 4.1(c)). This contradicts with Proposition 4.1(a).

Step 2: Now consider the case when M is seminormal and $n(M)$ is not free. By Propositions 2.2(d) and 2.4 we have $n(M)_* = M_*$. Consider the commutative diagram

$$\begin{tikzcd}
SK_1(R[M]) \arrow[rr] & & SK_1(R[n(M)]) \\
& SK_1(R[M_*]) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

By the previous step there exists $[A] \in SK_1(R[M_*])$, the image of which in $SK_1(R[n(M)])$ does not belong to $SK_1(R)$. It is clear from the diagram above that the image of $[A]$ in $SK_1(R[M])$ also does not belong to $SK_1(R)$.

Step 3: It remains to consider the case when M is seminormal and $n(M) = \mathbb{Z}_+^2$. By Propositions 2.2(d) and 2.4 we easily see that there exist natural numbers a and b, for which the following conditions are satisfied:

(i) $\max(a, b) \geq 2$,

(ii) $M = \mathbb{Z}_+^2 \setminus (\{X^i \mid a \text{ does not divide } i\} \cup \{Y^j \mid b \text{ does not divide } j\})$.

Arguing as in Step 1 we see that there exists a preimage $z \in St(R[X, Y])$ of $\langle u, v\rangle \in K_2(R[u, v])$ the image A of which in $E(R[X, Y])$ belongs to

$SL(R[M_*]) \cap SL(R[X,Y],(XY))$. Let $[A]$ denote the image of A in $SK_1(R[M])$. We claim that $[A] \notin SK_1(R)$. Suppose $[A] \in SK_1(R)$. Then (as in Step 1) we would have $A \in E(R[M])$. Let s be any lifting of $A \in E(R[M])$ in $St(R[M])$. We have the following natural commutative square:

$$\begin{tikzcd} R[M] \arrow[r] \arrow[d] & R[X,Y] \arrow[d] \\ R[u^a, v^b] \arrow[r] & R[u,v] \end{tikzcd}$$

Denote by s_0 the image of s in $St(R[u^a,v^b])$, by s_1 the image of s in $St(R[X,Y]$ and by s_2 the image of s_1 in $St(R[u,v])$. Since both s_1 and z map into $A \in E(R[M])$ we have $s_1 z^{-1} \in K_2[X,Y] = K_2(R)$. Therefore, $s_2\langle u,v\rangle^{-1} \in K_2(R)$. Again, modifying s (as in Step 1) we can assume that $s_1 = z$ and $s_2 = \langle u,v\rangle$. On the other hand, s_2 is the image of s_0 under the map $St(R[u^a,v^b]) \to St(R[u,v])$. Since the imbedding $R[u^a,v^b] \rightarrowtail R[u,v]$ the image of s_0 in $E(R[u^a,v^b])$ must coincide with 1. Hence $s_0 \in K_2(R[u^a,v^b])$. By Theorem 4.2, $s_0 = \langle u^a,v^b\rangle r \oplus p$ for some $r \in R$ and $p \in K_2(R)$ (here $\langle u^a,v^b\rangle = [x_{12}(u^a), x_{21}(v^b)] \in K_2(R[u^a,v^b])$). Clearly, we used the fact that $R[u^a,v^b]$ itself is of type $R[u,v]$, i.e. it is naturally isomorphic to $R[u,v]$. By Proposition 4.1(b), the element $\langle u^a,v^b\rangle r \oplus p \in K_2(R[u^a,v^b])$ maps into p under the map $K_2(R[u^a,v^b]) \to K_2(R[u,v])$, which is induced by the inclusion $R[u^a,v^b] \rightarrowtail R[u,v]$. Thus, we obtain $\langle u,v\rangle \in K_2(R)$, a contradiction by Proposition 4.1(a).

6. Quasifree monoids

Definition 6.1. A monoid M will be called quasifree if it is Φ-simplicial, seminormal and for any codimension 1 face d of $\Phi(M)$ the monoid $M(d)$ is free.

Obviously, free monoids are quasifree. It is also clear from Proposition 2.2 that a finitely generated rank 2 monoid M is quasifree iff M is semi-normal and $U(M)$ is trivial.

Proposition 6.2. *For a quasifree monoid M and a K_1-regular ring R the following conditions are equivalent:*

(a) *the image of the natural homomorphism $SK_1(R[M_*]) \to SK_1(R[M])$ is strictly larger than $SK_1(R)$,*

(b) *the homomorphism $SK_1(R[M_*]) \to SK_1(R[M])$ does not pass through $SK_1(R)$, i.e. the diagram*

$$\begin{tikzcd} & SK_1(R) \arrow[dr] & \\ SK_1(R[M_*]) \arrow[ur, "SK_1(\pi)"] \arrow[rr] & & SK_1(R[M]) \end{tikzcd}$$

where $M_* \xrightarrow{\pi} 0 \in R$, *does not commute,*

(c) *the natural homomorphism* $SK_1(R) \to SK_1(R[M])$ *is not an isomorphism.*

In the proof we shall use

Proposition 6.3. *Let* R *be a* K_1*-regular ring and* F *a free monoid. Then the natural homomorphism* $SK_1(R) \to SK_1(R[F]/(\mathrm{int}(F)))$ *is an isomorphism (here* $(\mathrm{int}(F))$ *denotes the ideal of* $R[F]$, *generated by* $\mathrm{int}(F)$).

This proposition is a special case from [14, Theorem 3.2.1(c)] (see also [22]).

Proof of Proposition 6.2. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) is clear. Now suppose $SK_1(R) \to SK_1(R[M])$ is not an isomorphism and, at the same time, the diagram

$$\begin{tikzcd} & SK_1(R) \arrow[dr] & \\ SK_1(R[M_*]) \arrow[ur] \arrow[rr] & & SK_1(R[M]) \end{tikzcd}$$

commutes. Consider the following Cartesian square with surjective vertical maps

$$\begin{tikzcd} R[M_*] \arrow[r] \arrow[d] & R[M] \arrow[d] \\ R \arrow[r] & R[M]/(\mathrm{int}(M)) \end{tikzcd}$$

We obtain the exact sequence

$$SK_1(R[M_*]) \to SK_1(R) \oplus SK_1(R[M]) \to SK_1(R[M]/(\mathrm{int}(M))).$$

Let $P_1, \ldots, P_r$ be the vertices of $\Phi(M)$ ($r = \mathrm{rank}(M)$). It is clear from Definition 6.1 that the inclusion map $F \rightarrowtail M$ induces the R-isomorphism

$$R[F]/(\mathrm{int}(F)) \approx R[M]/(\mathrm{int}(M)),$$

where F is the free submonoid of M generated by $M(P_1) \cup \cdots \cup M(P_r)$. By Proposition 6.3 we get the exact sequence

$$SK_1(R[M_*]) \longrightarrow SK_1(R) \oplus SK_1(R[M]) \longrightarrow SK_1(R).$$

Hence $SK_1(R[M_*]) \to SK_1(R[M])$ is onto. In this situation the aforementioned commutative diagram implies that $SK_1(R) \to SK_1(R[M])$ is onto, or equivalently $SK_1(R) = SK_1(R[M])$, a contradiction.

Below we shall use the fact that K_i-regularity (for some ring) implies K_{i-1}-regularity [23]. In particular, K_2-regular rings are K_1-regular as well.

Proposition 6.4. *Let M be a quasifree nonfree monoid and R a K_2-regular ring. Then the natural homomorphism $SK_1(R) \to SK_1(R[M])$ is not an isomorphism.*

Proof. This will be carried out by induction on $r = \operatorname{rank}(M)$. For $r = 1$ there is nothing to prove: there merely do not exist quasifree nonfree rank 1 monoids. For $r = 2$ we can apply Proposition 5.1 due to Proposition 6.2. Now assume that $\operatorname{rank}(M) = r \geq 3$ and 6.4 is proved for rank $r - 1$ quasifree nonfree monoids. We want to show that $SK_1(R) \neq SK_1(R[M])$. Suppose $SK_1(R) = SK_1(R[M])$.

Step 1: Let $P_1, \dots, P_r$ be the vertices of $\Phi(M)$. Consider the normalization $N = n(M)$. By Proposition 2.6 there exists a submonoid $L \subset N$, satisfying the conditions: L is normal in N, $L_* \subset N_*$ and $N(P_1)^{-1}N = K(N(P_1)) \oplus L$. By Propositions 2.2(a) and 2.4, $M_* = N_*$. We have the sequence of maps

$$L_* \rightarrowtail N_* = M_* \rightarrowtail M \rightarrowtail N \rightarrowtail K(N(P_1)) \oplus L \to L,$$

where the last map is a projection on L. Clearly, the corresponding composite map coincides with the inclusion map $L_* \rightarrowtail L$. Consider the composite map $M \rightarrowtail N \rightarrowtail K(N(P_1)) \oplus L \to L$. We easily obtain the commutativity of the following diagram:

$$\begin{array}{ccc} M & \longrightarrow & L \\ \uparrow & & \uparrow \\ M_* & \longrightarrow & L_* \end{array} \tag{$*$}$$

Let d be any proper face (of arbitrary positive dimension) of $\Phi(M)$, passing through P_1. Again by an easy geometrical observation we see that the following diagram commutes:

$$\begin{array}{ccc} M & \longrightarrow & L \\ \uparrow & & \uparrow \\ M(d) & \longrightarrow & L(d') \end{array} \tag{$**$}$$

where $d' = d \cap \Phi(L)$ is the corresponding face (of dimension $\dim(d) - 1$) of the simplex $\Phi(L)$.

Step 2: Let L' denote the image of M in L. We already know by $(*)$ that $L_* \subset L'$. We claim that L' is a quasifree monoid, for which $\Phi(L') = \Phi(L)$. Thus we have to show that the vertices of $\Phi(L)$ belong to $\Phi(L')$ and that for any codimension 1 face d' of $\Phi(L)$ the monoid $L'(d')$ is free. Observe that in this situation L' is automatically seminormal as a union of seminormal (even normal) monoids L_* (recall that L is normal) and $L'(d')$ (d' varies over codimension 1 faces of $\Phi(L)$). Since L' is finitely generated $\Phi(L')$ contains the closure $\Phi(L)$ of $\Phi(L'_*) = \Phi(L_*)$. Hence the vertices of $\Phi(L)$ actually belong to $\Phi(L')$. Now let d' be any codimension 1 face of $\Phi(L)$. Denote by d the uniquely determined codimension 1 face of $\Phi(M)$ passing through P_1, for which

$d' = d \cap \Phi(L)$. Since M is quasifree $M(d) = \bigoplus_{i \neq i_0} M(P_i)$ for some $i_0 \in [2, r]$, where each $M(P_i)$ is isomorphic to $\mathbb{Z}_+$. The image of $M(d)$ in L will be generated by the images of $M(P_i)$ for $i \neq 1, i_0$. Clearly, these images are also isomorphic to $\mathbb{Z}_+$ and the corresponding generators are linearly independent (in $K(L(d'))$). In other words $L'(d')$ is free. Having established the inclusion $L_* \subset L'$, the equality $\Phi(L') = \Phi(L)$ and the quasifreeness of L', now we are going to show that $L' = L$. Suppose $L' \neq L$ and consider the commutative diagram

$$\begin{tikzcd} & SK_1(R[M]) \arrow[dr] & \\ SK_1(R[L_*]) \arrow[ur] \arrow[rr] & & SK_1(R[L']) \end{tikzcd}$$

Since by our assumption $SK_1(R) = SK_1(R[M])$ we obtain the commutative diagram

$$\begin{tikzcd} & SK_1(R) \arrow[dr] & \\ SK_1(R[L_*]) \arrow[ur] \arrow[rr] & & SK_1(R[L']) \end{tikzcd}$$

Since $K(L') = K(L_*) = K(L)$ (by Proposition 2.2(b)) L' cannot be normal: otherwise L' must coincide with L. In particular, L' is not free. By induction hypothesis and Proposition 6.2 the diagram above cannot commute (here we use the equality $L'_* = L_*$). This contradiction implies that $L' = L$.

The very same arguments we used above show that L is free.

Thus we come to the conclusion that the assumption $SK_1(R) = SK_1(R[M])$ implies that $M \to L$ is onto and L is free.

Step 3: Now we shall show that the arguments above imply that M is free, contradicting the assumption of the proposition that M is not free.

By Proposition 2.2(f) we can assume that $M \subset \mathbb{Z}_+^r$ and that the extension is integral. We shall identify $\mathbb{Z}_+^r$ with the multiplicative monoid of monomials

$$\{t_1^{a_1} \cdots t_r^{a_r} \mid a_1, \ldots, a_r \in \mathbb{Z}_+\}.$$

It can be also assumed that $\Phi(t_i) = P_i$ for $i \in [1, r]$. Let $t_i^{a_i}$ be the generators of $M(P_i) \approx \mathbb{Z}_+$ and $t_i^{b_i}$ those of $N(P_i) \approx \mathbb{Z}_+$, respectively. Clearly, b_i divides a_i for each $i \in [1, r]$ (because $M(P_i) \subset N(P_i)$). Assume $\{T_2, \ldots, T_r\}$ is a basis of $L \approx \mathbb{Z}_+^{r-1}$. We can assume that $\Phi(T_i)$ belongs to the segment spanned by $\Phi(t_1)$ and $\Phi(t_i)$ ($i \in [2, r]$). We easily conclude that $\Phi(T_i)$ cannot coincide with $\Phi(t_1)$ for $i \in [2, r]$. It easily follows from the equality $N(P_1)^{-1} N = K(N(P_1)) \oplus L$ that the group $K(N(d_i))$ is generated by $t_1^{b_1}$ and T_i (for each $i \in [2, r]$), where d_i is the segment spanned by $\Phi(t_1)$ and $\Phi(t_i)$. In particular,

$$t_i^{b_i} = T_i^{x_i} \cdot (t_1^{b_1})^{-y_i}, \quad i \in [2, r]$$

for some strictly positive integers x_i and $y_i \in \mathbb{Z}_+$ (since $\Phi(T_i)$ cannot coincide with $\Phi(t_1)$ for $i \in [2, r]$). Put $c_i = a_i/b_i$. We get

$$t_i^{a_i} = T_i^{c_i x_i} \cdot (t_1^{b_1})^{-c_i y_i}, \quad i \in [2, r].$$

As we have shown L is freely generated by the images of $t_i^{a_i}$'s in L. But these images are just $T_i^{c_i x_i}$'s. Since a free monoid has a unique basis we conclude that $c_i x_i = 1$, or equivalently $c_i = x_i = 1$ for $i \in [2, r]$. Finally, we have (for each $i \in [2, r]$) $a_i = b_i$ and T_i belongs to the monoid generated by $t_1^{b_1}$ and $t_i^{a_i}$. By symmetry (we can use the same arguments) $a_1 = b_1$ as well. Since $N = n(M)$ the groups $K(N)$ and $K(M)$ coincide. Consequently, the objects M, P_1 and F, where F denotes the submonoid of N generated by $M(P_1) = N(P_1)$ and L, satisfy the conditions of Proposition 2.8 (stated for M, P and F, respectively). Hence M is free – the desired contradiction.

7. Main theorem

Theorem 7.1. *Let R be a regular (or even K_2-regular) ring and M a Φ-simplicial monoid. Then*

(a) *$R[M]$ is K_1-regular if and only if M is free,*

(b) *M is seminormal and $SK_1(R) = SK_1(R[M])$ if and only if M is free;*

(c) *If Ω_R is not trivial then $SK_1(R) = SK_1(R[M])$ if and only if M is free.*

We shall need the following two lemmas.

Lemma 7.2. *For any commutative ring R there is a surjective homomorphism from $K_2(R[X]/(X^2))/K_2(R)$ to Ω_R.*

Proof. This directly follows from [21], where it is shown that for a commutative ring A and its nilpotent ideal I the relative group $K_2(A, I)$ has a presentation by Dennis–Stein symbols (and their standard relations). We easily observe that in the special case $A = R[X]/(X^2)$ and $I = R\varepsilon$, where $\varepsilon = X + (X^2)$, if one factors out the extra relation $\langle r\varepsilon, s\varepsilon \rangle = 0$, r, $s \in R$, the Dennis–Stein presentation becomes precisely the definition of Ω_R. So there always is a surjection $K_2(R[X]/(X^2))/K_2(R) \to \Omega_R$ and its kernel is generated by Dennis–Stein symbols $\langle r\varepsilon, s\varepsilon \rangle$. $\square$

Lemma 7.3. *Let R be K_2-regular ring and X a variable. Then there is a surjective homomorphism from $SK_1(R[X^2, X^3])/SK_1(R)$ to Ω_R.*

Proof. This is proved in [15, Lemma 12.1] in the case when R is a field. However, we shall show that Krusemeyer's arguments imply the mentioned general case. Since R is K_2-regular by [23] it is K_1-regular too. In particular, R is reduced. In this situation

$K_1(R[X], (X^2)) = SK_1(R[X], (X^2))$. Then by [18, Theorem 15.4], we obtain the exact sequence

$$K_2(R[X]) \to K_2(R[X]/(X^2)) \to SK_1(R[X],(X^2)) \to K_1(R[X])$$

(compare with [15, p. 37]). By K_1-regularity of R we have $K_1(R[X]) = K_1(R)$. Therefore, $SK_1(R[X],(X^2)) \to K_1(R[X])$ is a zero map. Thus, we obtain $SK_1(R[X],(X^2)) = K_2(R[X]/(X^2))/K_2(R)$. Now we are done by Lemma 7.2 and the observation that $SK_1(R[X^2,X^3])/SK_1(R)$ naturally maps onto $SK_1(R[X], (X^2))$. $\square$

Proof of Theorem 7.1. Let us first show that (b) $\Rightarrow$ (a). Since we have the implications (K_1-regularity) $\Rightarrow$ (K_0-regularity) $\Rightarrow$ (Pic-regularity) $\Leftrightarrow$ (seminormality in the sense of [20]), we easily deduce the seminormality of M from the K_1-regularity of $R[M]$ (see also [11, Appendeix]). Since the embedding $M \subset \mathbb{Z}^r_+$, $r = \operatorname{rank}(M)$, we have a grading $R[M] = R \oplus R_1 \oplus R_2 \oplus \cdots$ Now, by Weibel's well-known homotopic trick ([see, for instance [2, Lemma 5.7]) the equality $SK_1(R[M]) = SK_1(R[M][X])$ implies that $SK_1(R) = SK_1(R[M])$ (X is a variable). The "*if*" part of Theorem 7.1(a) follows from [6].

Now consider (b). The case $r = \operatorname{rank}(M) = 1$ is trivial (M is necessarily isomorphic to $\mathbb{Z}_+$). Suppose $r \geq 2$. Let d be any codimension 1 face of $\Phi(M)$. Since $R[M(d)]$ is an R-retract of $R[M]$ ($R[M(d)] \leftrightarrows R[M]$, $M \backslash M(d) \to 0 \in R$), we have $SK_1(R) = SK_1(R[M(d)])$. By induction hypothesis $M(d)$ is free. Hence M is quasifree and Proposition 6.4 applies. The "*if*" part of Theorem 7.1(b) follows from [6].

(c) *Step* 1: $r = \operatorname{rank}(M) = 1$. The monoid operation will be written additively. Clearly, without loss of generality, we can assume that $M \subset \mathbb{Z}_+$ and $K(M) = \mathbb{Z}$. By Proposition 2.9 there exists $m \in M^+$ such that $m + \mathbb{Z}_+ \subset M$ (in our situation $\operatorname{int}(M) = M^+$). We want to show that $M = \mathbb{Z}_+$. Suppose $M \neq \mathbb{Z}_+$. Then M is contained in $\{0,2,3,\dots\}$. Both $R[M]$ and $R[X^2,X^3]$ contain the ideal $I = (X^m, X^{m+1}, X^{m+2}, \dots)$ (clearly, $R[\mathbb{Z}_+]$ is identified with $R[X]$). Since $R[M]/I$ and $R[X^2,X^3]/I$ are nilpotent extensions of R the Mayer–Vietoris sequence yields a surjection $SK_1(R[M]) \to SK_1(R[X^2,X^3])$. By Lemma 7.3 the latter is larger than $SK_1(R)$.

Step 2: Assume $r = \operatorname{rank}(M) \geq 2$, $SK_1(R) = SK_1(R[M])$ and Theorem 7.1(c) is proved for rank $r-1$ Φ-simplicial monoids. By the induction hypothesis for any codimension 1 face d of $\Phi(M)$ the monoid $M(d)$ is free (see the proof of Theorem 7.1(b). We claim that $SK_1(R) = SK_1(R[sn(M)])$. By Proposition 2.9 there exists $m \in \operatorname{int}(M)$ for which $m + n(M) \subset M$ (writing additively). Put $I = mR[sn(M)]$. Then I will be an ideal of both $R[M]$ and $R[sn(M)]$. One also easily observes that $(R[M]/I)_{\text{red}} = R[F]/(\operatorname{int}(F)) = (R[sn(M)]/I)_{\text{red}}$ for F the free submonoid of M generated by $M(P_1) \cup \cdots \cup M(P_r)$, where $P_1, \dots, P_r$ are the vertices of $\Phi(M)$, so $SK_1(R[M]/I) = SK_1(R[sn(M)/I)$. By the Mayer–Vietoris sequence $SK_1(R[M]) \to SK_1(R[sn(M)])$ is onto. Thus, $SK_1(R) = SK_1(R[sn(M)])$. By Theorem 7.1(b), $sn(M)$ must be free. On the other hand, for any codimension 1 face of $\Phi(M)$ we have

$sn(M)(d) = M(d)$ (recall that $M(d)$ is free). This implies that the base's of the free monoids F and $sn(M)$ coincide. Hence, $F = sn(M)$. Now the inclusions $F \subset M \subset sn(M) = F$ complete the proof. □

8. Examples

The arguments from Lemma 7.2 show that for a rather restricted class of rings R, such that $\Omega_R = 0$ and the Dennis–Stein symbols mentioned in this lemma are trivial (for instance if $\frac{1}{2} \in R$), we obtain examples of monoid rings $R[M]$, which are not K_1-regular, but $K_1(R) = K_1(R[M])$.

Example 8.1 (Krusemeyer, [15 p. 37]). Let R be a number field (i.e. an algebraic extension of $\mathbb{Q}$) and X a variable. Then $R[X^2, X^3]$ is not K_1-regular, but $K_1(R) = K_1(R[X^2, X^3])$.

Example 8.2 (Srinivas' example). Consider the following monoid algebra $R[X^2, XY, Y^2]$, where X and Y are variables. This algebra is isomorphic to $R[Y, XY, YX^2]$ ($X^2 \mapsto Y$, $XY \mapsto XY$, $Y^2 \mapsto YX^2$), which is a monoid R-algebra, corresponding to the submonoid $M \subset \mathbb{Z}_+^2$ generated by $\{(0,1),(1,1),(2,1)\}$. It can be easily checked that the embedding $M \rightarrowtail \mathbb{Z}_+^2$ satisfies conditions (i) and (ii) from 3.2. Let A_1 be the same matrix as in the proof of Proposition 4.3 (with respect to R). Then $A_1 \in SL_2(R[L])$ and Step 1 in the proof of Proposition 5.1 shows that in case R is K_2-regular $[A_1] \in SK_1(R[L]) \backslash SK_1(R)$. Let A be the matrix obtained from A_1 by permuting rows and columns. Then $[A] = [A_1]$ in $SK_1(R[L])$.

Let us recall that for a commutative ring D and $a, b \in D$, such that $aD + bD = D$, there is a Mennicke symbol $[a, b] \in SK_1(D)$ determined as follows: choose $c, d \in D$ so that $ad - bc = 1$, then

$$[a, b] = \text{class of} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ in } SK_1(D).$$

The point is that the class in $SK_1(D)$ is independent of the choices [17].

Returning to our example we obtain that

$$[1 - XY,\ XY^2] \in SK_1(R[Y, XY, YX^2]) \backslash SK_1(R),$$

or equivalently

$$[1 - XY, X^2] = [1 - XY, X^3Y] \in SK_1(R[X^2, XY, Y^2]) \backslash SK_1(R).$$

In the particular case when R is an algebraically closed field of characteristic $\neq 2$ we obtain the example of [17, Section 4].

9. Further generalizations

Our results can be improved in two directions: the first one is to answer the question raised in the introduction and the second one concerns the explicit description of $SK_1(R[M])$.

At first let us mention some results toward the positive answer to the aforementioned question, which can be obtained by the methods we presented above.

Theorem 9.1. *Let R be a K_2-regular ring. Then*

(a) $SK_1(R) \neq SK_1(R[M])$ *for arbitrary finitely generated normal nonfree monoids M with trivial $U(M)$ whenever $SK_1(R) \neq SK_1(R[M])$ for finitely generated normal nonfree monoids M with trivial $U(M)$ for which* $\operatorname{rank}(M) \leq 3$;

(b) $SK_1(R) \neq SK_1(R[M])$ *whenever M is a rank r submonoid of $\mathbb{Z}$ for some natural r, such that the following conditions are satisfied:*

(i) *$n(M)$ is normal in $\mathbb{Z}^r_+$,*

(ii) *there exists an element $(a_1, \dots, a_r) \in \operatorname{int}(\mathbb{Z}^r_+) \cap \operatorname{int}(M)$, such that $a_i = a_j = 1$ for some $i \neq j$,*

(iii) *$\Phi(M) \cap \partial\Phi(\mathbb{Z}^r_+)$ is contained in some codimension 1 face of $\Phi(M)$, where $\partial\Phi(\mathbb{Z}^r_+)$ denotes the boundary of the simplex $\Phi(\mathbb{Z}^r_+)$;*

(c) *In order to show that $SK_1(R) \neq SK_1(R[M])$ for arbitrary finitely generated nonfree monoids M with trivial $U(M)$ it suffices to show that the natural maps $SK_1(R[M_*]) \to SK_1(R[M])$ do not pass through $SK_1(R)$ for finitely generated normal nonfree monoids M with trivial $U(M)$; furthermore, in the situation described in* (b), *the natural map $SK_1(R[M_*]) \to SK_1(R[M])$ do not pass through $SK_1(R)$.*

Sketch of Proof. (a) is based on the following well-known result of combinatorial geometry: a finite convex polyhedron of dimension ≥ 3 is a simplex if its all codimension 1 faces are simplices and the polyhedral cones, spanned by this polyhedron at its vertices, are all simplicial (for details see, for instance, Arne Bröndsted, *An Introduction to Convex Polytopes*, Springer, New York (1983), Theorem 12.19). Indeed, assume that $SK_1(R) = SK_1(R[M])$ for some finitely generated normal monoid M with trivial $U(M)$. If $\Phi(M)$ is a simplex then M is free by Theorem 7.1. If $\Phi(M)$ is not a simplex and $\operatorname{rank}(M) = \dim(\Phi(M)) + 1 \geq 4$ then either some of codimension 1 faces, say d, of $\Phi(M)$ is not a simplex or the polyhedral cone spanned by $\Phi(M)$ at some of its vertices, say P, is not simplicial. Correspondingly, in the first case we obtain the equality $SK_1(R) = SK_1(R[M(d)])$ and in the second one the equality $SK_1(R) = SK_1(R[M'])$, where M' is the same for M (with respect to P) as in Proposition 2.6; the crucial point here is that $R[M(d)]$ and $R[M']$ are both retracts of $R[M]$. Since neither $M(d)$ nor M' (in the corresponding case) is not Φ-simplicial the induction process on $\operatorname{rank}(M)$ applies.

(b) First observe that in case $r = 2$ conditions (i)(iii) for $n(M)$ degenerate exactly into conditions (i) and (ii) of Proposition 3.2. Not describing the details we only mention here that the proof is parallel to the proof of Proposition 5.1. The starting point in our arguments is that

$$R[X,Y]/(XY) \approx R[t_i,f_i]/(f_it_i)$$

is an R-retract of $R[t_1,\dots,t_r]/(t_1\cdots t_r)$ via the maps

$$R[t_i,f_i]/(f_it_i) \rightarrowtail R[t_1,\dots,t_r]/(t_1\cdots t_r),$$

induced by the embedding

$$R[t_i,f_i] \rightarrowtail R[t_1,\dots,t_r]$$

and

$$R[t_1,\dots,t_r]/(t_1\cdots t_r) \to R[t_i,f_i]/(f_it_i),$$

induced by $t_i \mapsto t_i$, $t_j \mapsto f_i$ and $t_s \mapsto 1$ for $s \neq i,j$ (here $\mathbb{Z}^r_+$ is naturally identified with the multiplicative monoid of "pure" monomials in variables t_s), where $f_i = t_1^{a_1}\cdots t_{i-1}^{a_{i-1}}\hat{t}_i^{a_i}t_{i+1}^{a_{i+1}}\cdots t_r^{a_r}$.

We remark that the obstruction to the final general result is the lack of the direct analogoue of Proposition 3.2 for higher rank monoids. The normal submonoid $\mathbb{Z}_+(1,0,0)+\mathbb{Z}_+(0,1,0)+\mathbb{Z}_+(1,0,1)+\mathbb{Z}_+(0,1,1)\subset\mathbb{Z}^3_+$ is the example of such a nonfree finitely generated normal monoid without nontrivial units, which cannot be embedded in $\mathbb{Z}^3_+$ so that conditions (i)–(iii) of Theorem 9.1(b) will be satisfied.

(c) can be proved by the arguments we used in Step 2 of the proof of Theorem 7.1.

In conclusion we only mention that it is not clear whether even for Φ-simplicial normal nonfree monoids M the natural homomorphism $SK_1(R[M_*]) \to SK_1(R[M]$ does not pass through $SK_1(R)$.

As it was mentioned above it would be interesting to have some reasonable description of $SK_1(R[M])/SK_1(R)$ (even for rank 2 normal monoids). At this moment we cannot suggest any concrete conjecture in this direction. The following proposition might serve here as a starting point.

Proposition 9.2. *Let R be a K_2-regular ring and M a Φ-simplicial monoid.*

(a) *Assume $c \geq 2$ is a natural number and consider the map $c_{R,M}: R[M] \to R[M]$ induced by $m \mapsto m^c$, $m \in M$ (writing multiplicatively). Then all the elements in $SK_1(R[M])$ representing nonzero elements in $SK_1(R[M])/SK_1(R)$, which were constructed in this paper, trivialize under the corresponding map $SK_1(c_{R,M})$: $SK_1(R[M]) \to SK_1(R[M])$;*

(b) *If $\mathbb{Z}\cdot 1 \approx \mathbb{Z}$ for $1 \in R$ and M is seminormal then the nontrivial elements in $SK_1(R[M])/SK_1(R)$, we constructed in this paper, are of infinite order.*

Proof. The proof of these statements easily follows from Proposition 4.1(b), Theorem 4.2 and the appropriate adaptation of the proof of Theorem 7.1. Here is an alternative proof of the claim (b): if R contains $\mathbb{Q}$ then the group $SK_1(R[M])/SK_1(R)$ is uniquely divisible, because $W(R)$ contains $\mathbb{Q}$ (see the remark after Theorem 4.2 above), so of course all nontrivial elements have infinite order; if R contains $\mathbb{Z}$ the elements we construct remain nontrivial over the localization $R_{\mathbb{Q}}$, so this implies that they have infinite order already.

It seems very probable that the hypothetical description of $SK_1(R[M])/SK_1(R)$ will involve the divisor class group $Cl(M)$ (measuring the deviation from the freeness, see [14]), $\operatorname{rank}(M)$ and some invariants of R.

10. A connection with homological properties

Proposition 10.1. *For a regular ring R and a finitely generated monoid M with trivial $U(M)$ the monoid ring $R[M]$ is regular iff M is free.*

Proof. The proof reduces easily (by localization) to the case when R is a field. In this situation $\mu = R \cdot M^+ \in \max(R)$. Since all local regular rings are factorial so must be $R[M]_\mu$. But it can be checked very easily that $R[M]_\mu$ is factorial iff any element of M admits a unique factorization into indecomposable elements, or equivalently iff M is free. □

The results of this paper show that for a rather wide class of finitely generated monoids without nontrivial units even the K_1-regularity of the corresponding monoid ring implies the freeness of monoids. Basing on these results we suggest one homological criterion for M to be free. For these purposes suppose R is a ring, M a finitely generated monoid with trivial $U(M)$, $c \geq 2$ and $c_{R,M}: R[M] \to R[M]$, the R-homomorphism, induced by $m \mapsto m^c$, $m \in M$ (writing multiplicatively). Denote by $pd(c_{R,M})$ the projective dimension of $R[M]$, which is considered as an $R[M]$-module via $c_{R,M}$.

Proposition 10.2. *Assume that $c \geq 2$, R is a domain of characteristic 0 and M is a Φ-simplicial seminormal monoid. Then $pd(c_{R,M}) < \infty$ if and only if M is free and in this case $pd(c_{R,M}) = 0$.*

Proof. If M is free then $c_{R,M}$ makes $R[M]$ a free $R[M]$-module (of rank $c^{\operatorname{rank}(M)}$) and hence $pd(c_{R,M}) = 0$. Now suppose $pd(c_{R,M}) < \infty$. Since a localization functor is exact we can pass to the fraction field of R, in other words we can assume that R is a field. Put

$$c^{-1}M = \varinjlim \left(M \xrightarrow{(-)^c} M \xrightarrow{(-)^c} \cdots\right).$$

By [11, Theorem 2.1], $SK_1(R[c^{-1}M]) = 0$. Consider the diagram

$$K_1(R[M]) \xrightarrow{K_1(c_{R,M})} K_1(R[M] \xrightarrow{K_1(c_{R,M})} \cdots (D).$$

We have $\varinjlim (D) = K_1(R[c^{-1}M]) = K_1(R)$. Since $pd(c_{R,M}) < \infty$ there exists a transfer map $t_{c,R,M}: K_1(R[M]) \to K_1(R[M])$, such that the composite map $t_{c,R,M} \cdot K_1(c_{R,M})$ is a multiplication by the Euler characteristic $\chi(R[M]) \in K_0(R[M])$, where $R[M])$ is considered as an $R[M]$-module via $c_{R,M}$ (see [5, Ch. IX, Section 1, Proposition 1.8]). Let us show that $\chi(R[M]) = c^{\operatorname{rank}(M)}$ in $K_0(R[M]) = \mathbb{Z}$. Indeed, $K_0(R[M]) = \mathbb{Z}$ by [10]. Thus $\chi(R[M]) = \sum_{i=0}^{n} (-1)^i r_i$, where r_i-s are the ranks of $P_i \in \mathbb{P}(R[M])$ for some finite projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to R[M] \to 0.$$

Consider the localization with respect to the multiplicative subset $M \subset R[M]$. The aforementioned remarks imply that

$$\chi_{R[M]}(R[M]) = \chi_{R[K(M)]}(R[K(M)]),$$

where on the right-hand side $R[K(M)]$ is considered as an $R[K(M)]$-module via $c_{R,K(M)}: R[K(M)] \to R[K(M)]$. Since $K(M) \approx \mathbb{Z}^{\operatorname{rank}(M)}$ we obtain $\chi_{R[K(M)]}(R[K(M)]) = c^{\operatorname{rank}(M)}$. Now suppose M is not free. Then by Proposition 9.2(b) there exists an element $x \in SK_1(R[M])$ of infinite order. On the other hand, the aforementioned remarks imply that $K_1(c_{R,M})^m(x) = 0$ for some natural m. Using the map $t_{c,R,M}$ we get $c^{m \cdot \operatorname{rank}(M)}\, x = 0$, a contradiction. $\square$

Remarks. We may expect that Proposition 10.2 holds for all rings R and all finitely generated monoids M without nontrivial units. Actually our proof of Proposition 10.2 applies to the monoids, mentioned in Theorem 9.1(b).

In view of Proposition 9.2(a) it is natural to ask whether $K_1(c_{R,M})$ passes through $K_1(R)$ for a regular ring R and a finitely generated monoid M without nontrivial units.

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